14: Adding and Subtracting Rational Expressions

Adding rational expressions by finding common denominators

To add two fractions, we first find a common denominator. To do that we multiply the first fraction by d_2/d_2, where d_2 is the denominator of the second fraction. Then, we multiply the second fraction by d_1/d_1, where d_1 is the denominator of the first fraction. Now we have new common denominators equal to $d_1 d_2$ and we can simply add the new numerators. For example, $\frac{3}{5}+\frac{1}{3}=\frac{3}{5}\left(\frac{3}{3}\right)+\left(\frac{5}{5}\right)\frac{1}{3}=\frac{9}{15}+\frac{5}{15}=\frac{14}{15}$.

We do a similar thing when adding rational expressions. For example, $\frac{2}{x+1}+\frac{x}{x-2}=\frac{2}{x+1}\left(\frac{x-2}{x-2}\right)+\left(\frac{x+1}{x+1}\right)\frac{x}{x-2}=\frac{2x-4+x^2+x}{(x+1)(x-2)}=\frac{x^2+3x-4}{(x+1)(x-2)}=\frac{(x+4)(x-1)}{(x+1)(x-2)}, \{\forall\ x\neq -1, 2\}$.

Subtracting rational expressions

Similarly, we find common denominators first when subtracting rational expressions. For example, $\frac{x}{4x-1}-\frac{1}{x+2}=\frac{x}{4x-1}\left(\frac{x+2}{x+2}\right)-\left(\frac{4x-1}{4x-1}\right)\frac{1}{x+2}=\frac{x^2+2x-4x+1}{(4x-1)(x+2)}=\frac{x^2-2x+1}{(4x-1)(x+2)}=\frac{(x-1)^2}{(4x-1)(x+2)}, \left\{\forall\ x\neq -2, \frac{1}{4}\right\}$.

Simplifying using factoring

As previously, always look for opportunities to factor any of the polynomials before adding or subtracting. For example, $\frac{4}{x^2-2x+1}+\frac{3}{x-1}=\frac{4}{x^2-2x+1}\left(\frac{x-1}{x-1}\right)+\left(\frac{x^2-2x+1}{x^2-2x+1}\right)\frac{3}{x-1}$. This is not particularly easy to simplify from here.

However, if we factor the denominator of the first rational expression before adding, we get a much simpler answer: $\frac{4}{x^2-2x+1}+\frac{3}{x-1}=\frac{4}{(x-1)^2}+\frac{3}{x-1}=\frac{4}{(x-1)^2}+\left(\frac{x-1}{x-1}\right)\frac{3}{x-1}=\frac{4+3x-3}{(x-1)^2}=\frac{3x+1}{(x-1)^2}, \{\forall\ x\neq 1\}$. Note that the common denominator in this case is just $(x-1)^2$. There's no need to use $(x-1)^3$.

More examples

- $\frac{x+1}{2x}-\frac{4}{x^2+3x}=\frac{x+1}{2x}-\frac{4}{x(x+3)}=\frac{x+1}{2x}\left(\frac{x+3}{x+3}\right)-\left(\frac{2}{2}\right)\frac{4}{x(x+3)}=\frac{x^2+4x+3-8}{2x(x+3)}=\frac{x^2+4x-5}{2x(x+3)}$
  $=\frac{(x+5)(x-1)}{2x(x+3)}, \{\forall\ x\neq -3, 0\}$.
- $\frac{1}{x+2}+\frac{x}{x^2+3x+2}=\frac{1}{x+2}+\frac{x}{(x+2)(x+1)}=\frac{1}{x+2}\left(\frac{x+1}{x+1}\right)+\frac{x}{(x+2)(x+1)}=\frac{x+1+x}{(x+2)(x+1)}$
  $=\frac{2x+1}{(x+2)(x+1)}, \{\forall\ x\neq -2, -1\}$.
- $\frac{x-1}{x^2+x}-\frac{x-1}{2x^2}=\frac{x-1}{x(x+1)}-\frac{x-1}{2x^2}=\frac{x-1}{x(x+1)}\left(\frac{2x}{2x}\right)-\left(\frac{x+1}{x+1}\right)\frac{x-1}{2x^2}=\frac{2x^2-2x-x^2+1}{2x^2(x+1)}=\frac{x^2-2x+1}{2x^2(x+1)}$

$= \frac{(x-1)^2}{2x^2(x+1)}, \{\forall\ x \neq -1, 0\}.$

Alternative calculation:

- $\frac{x-1}{x^2+x} - \frac{x-1}{2x^2} = (x-1)\left[\frac{1}{x(x+1)} - \frac{1}{2x^2}\right] = (x-1)\left[\frac{1}{x(x+1)}\left(\frac{2x}{2x}\right) - \left(\frac{x+1}{x+1}\right)\frac{1}{2x^2}\right]$

  $= (x-1)\left[\frac{2x-x-1}{2x^2(x+1)}\right] = (x-1)\left[\frac{x-1}{2x^2(x+1)}\right] = \frac{(x-1)^2}{2x^2(x+1)}, \{\forall\ x \neq -1, 0\}.$