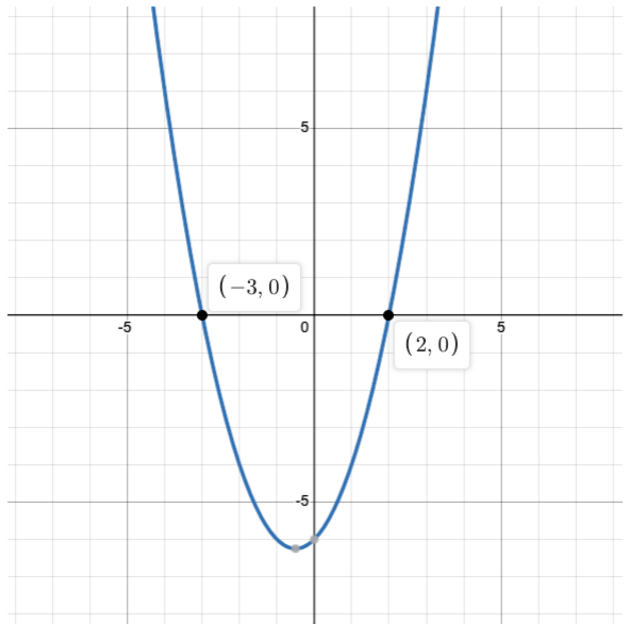
17: Solving Quadratic Equations by Factoring

Quadratic equations in one variable

Define a quadratic equation in one variable as $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. Solving this equation means to find all the values of x that satisfy this equation, i.e., the left-hand and right-hand sides of the equation are equal. This is also known as finding the *roots* of the equation.

Recall that the *zero-factor theorem* says that if a product of two numbers (factors) is 0, then one or both factors must be 0. Also recall that sometimes it is possible to factor a quadratic. We can put these two ideas together to easily solve *some* quadratic equations.

Two solutions



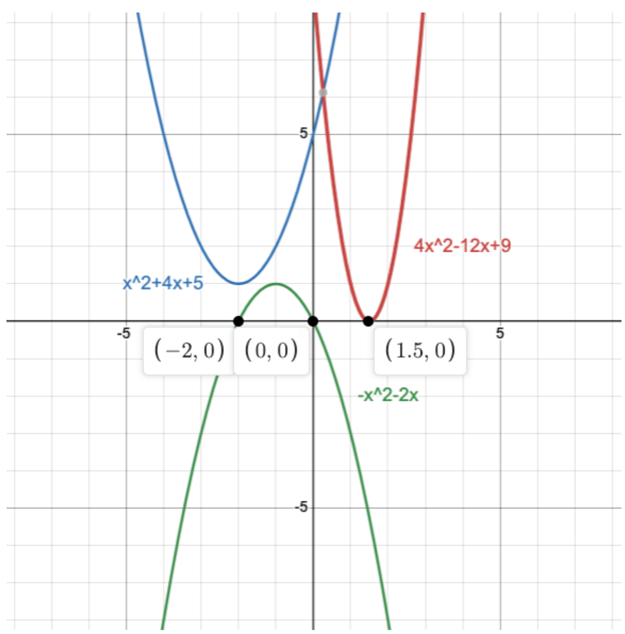
For example, $x^2 + x - 6 = (x+3)(x-2)$. So, if we want to solve the quadratic equation $x^2 + x - 6 = 0$, we can rewrite it as $(x+3)(x-2) = 0$, which has two solutions: $x = -3$ and $x = 2$.

As previously, it's a good idea to check these solutions to make sure they work and to catch any errors. In this case, $(-3)^2 + (-3) - 6 = 9 - 3 - 6 = 0$ and $(2)^2 + 2 - 6 = 4 + 2 - 6 = 0$, so the solutions $x = -3$ and $x = 2$ work.

The solutions are easy to see on a graph of the function $y = x^2 + x - 6$, to the right. The graph is a parabola opening upwards since the coefficient on x^2 is positive.

The solutions are the values of x when $y = 0$, where the graph crosses the x-axis in two places.

Two, one, or zero solutions



However, it's not always the case that there are two solutions. For example:

- $-x^2 - 2x = -x(x+2)$.

  Two solutions: $x = -2$, $x = 0$.

- $4x^2 - 12x + 9 = (2x-3)^2$.

  One solution: $x = \frac{3}{2}$.

- $x^2 + 4x + 5 = (x^2 + 4x + 4) + 1 = (x+2)^2 + 1$.
  No solutions.

More examples

- $-x^2 + 4 = 0 \Rightarrow -(x+2)(x-2) = 0 \Rightarrow x = -2, 2.$
- Two solutions: $x = -2, 2.$
- Check: $-(-2)^2 + 4 = -4 + 4 = 0$ and $-2^2 + 4 = -4 + 4 = 0.$
- $\frac{3}{2}x^2 + \frac{1}{4}x - \frac{15}{4} = 0 \Rightarrow \frac{1}{4}(6x^2 + x - 15) = 0 \Rightarrow \frac{1}{4}(3x+5)(2x-3) = 0 \Rightarrow x = -\frac{5}{3}, \frac{3}{2}.$
- Two solutions: $= -\frac{5}{3}, \frac{3}{2}.$
- Check: $\frac{3}{2}\left(-\frac{5}{3}\right)^2 + \frac{1}{4}\left(-\frac{5}{3}\right) - \frac{15}{4} = \frac{25}{6} - \frac{5}{12} - \frac{15}{4} = \frac{50}{12} - \frac{5}{12} - \frac{45}{12} = 0$

  and $\frac{3}{2}\left(\frac{3}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{2}\right) - \frac{15}{4} = \frac{27}{8} + \frac{3}{8} - \frac{15}{4} = \frac{27}{8} + \frac{3}{8} - \frac{30}{8} = 0.$