

19: Solving Linear Inequalities

Representing inequalities using interval notation

Thinking about numbers as points along a number line helps us understand inequalities. For example, the statement $x > 4$ indicates all real numbers that are greater than 4. We can denote this using interval notation as $x \in (4, \infty)$, where the parentheses (rounded brackets) indicate the endpoints of the interval are *not* included. We can visualize this on a number line as:



Other examples

- $x \geq 4$, i.e., $x \in [4, \infty)$. The left square bracket indicates the endpoint 4 is included.
- $x < 4$, i.e., $x \in (-\infty, 4)$. The parentheses indicate the endpoints are not included.
- $x \leq 4$, i.e., $x \in (-\infty, 4]$. The right square bracket indicates the endpoint 4 is included.
- $-4 < x < 4$, i.e., $x \in (-4, 4)$. The parentheses indicate the endpoints are not included.

Note that “infinity” (∞) and “negative infinity” ($-\infty$) are not real numbers and are always excluded from the endpoints of real number intervals.

Solving linear inequalities

To solve an inequality means to find all the values of x that satisfy the inequality, meaning that substituting one of those values for x would result in a true statement.

For example, solve $7x - 4 \leq 4x + 5$ by isolating x on one side of the inequality with just numbers on the other side. To do this we can add an expression to both sides of the inequality or multiply both sides of the inequality by an expression that is always positive. We can then check the answer with a test value.

- $7x - 4 + 4 \leq 4x + 5 + 4$
- $7x - 4x \leq 4x + 9 - 4x$
- $3x \times \frac{1}{3} \leq 9 \times \frac{1}{3}$
- $x \leq 3$, i.e., $x \in (-\infty, 3]$.
- Check: $7(2) - 4 \leq 4(2) + 5$: $10 \leq 13$.

We must be careful to reverse the inequality when multiplying both sides of the inequality by an expression that is always negative. For example: $1 < 2$ but $-1 > -2$.

An example of multiplying both sides by a negative expression

Find the values of x such that $3x + 2 < 5x + 6$.

- $3x + 2 - 2 < 5x + 6 - 2$
- $3x - 5x < 5x + 4 - 5x$

- $-2x \times \left(-\frac{1}{2}\right) > 4 \times \left(-\frac{1}{2}\right)$
- $x > -2$, i. e., $x \in [-2, \infty)$.
- Check: $3(-1) + 2 < 5(-1) + 6$: $-1 < 1$.

An example of a continued inequality (isolate x in the middle)

Find the values of x such that $-1 < \frac{-4x+3}{5} \leq 3$.

- $-5 < -4x + 3 \leq 15$
- $-8 < -4x \leq 12$
- $2 > x \geq -3 \Rightarrow -3 \leq x < 2$, i. e., $x \in [-3, 2)$.
- Check: $-1 < \frac{-4(0)+3}{5} \leq 3$: $-1 < \frac{3}{5} \leq 3$.

More examples

Find the values of x such that $9x + 2 \geq 5x - 2$.

- $9x - 5x \geq -2 - 2$
- $4x \geq -4$
- $x \geq -4 \times \frac{1}{4} = -1$, i. e., $x \in [-1, \infty)$.
- Check: $9(1) + 2 \geq 5(1) - 2$: $11 \geq 3$.

Find the values of x such that $-5 \leq \frac{-2x-5}{3} < 1$.

- $-15 \leq -2x - 5 < 3$
- $-15 + 5 \leq -2x < 3 + 5$
- $5 \geq x > -4 \Rightarrow -4 < x \leq 5$, i. e., $x \in (-4, 5]$.
- Check: $-1 < \frac{-4(0)+3}{5} \leq 3$: $-1 < \frac{3}{5} \leq 3$.