20: Solving Other Inequalities

Solving an inequality that involves a quadratic expression

To solve a linear inequality, we isolate x on one side of the inequality with just numbers on the other side. For example, $7x - 4 \leq 4x + 5$ simplifies to $x \leq 3$, i.e., $x \in (-\infty, 3]$. What about an inequality that involves a quadratic expression such as $x^2 + 4x - 3 \leq 5x + 9$?

- $x^2 + 4x - 3 - 5x - 9 \leq 0$
- $x^2 - x - 12 \leq 0$
- $(x + 3)(x - 4) \leq 0$

Since the quadratic on the lefthand side factors, we know that the roots are $x = -3$ and $x = 4$ and since the coefficient on x^2 is positive we know the parabola opens upwards. Therefore, the solution must be $-3 \leq x \leq 4$, i.e., $x \in [-3, 4]$.

We can check with test values:

- $(-4)^2 + 4(-4) - 3 \leq 5(-4) + 9 \Rightarrow -3 \leq -11$: false
- $1^2 + 4(1) - 3 \leq 5(1) + 9 \Rightarrow 2 \leq 14$: true
- $5^2 + 4(5) - 3 \leq 5(5) + 9 \Rightarrow 42 \leq 34$: false

Solving an inequality that involves a rational expression

What about an inequality that involves a rational expression such as $\frac{x+2}{x-1} \geq 0$?

- When $x = 1$, $y = \frac{x+2}{x-1}$ is undefined.
- When $x = -2$, $y = 0$.
- When $x < -2$, numerator is negative, denominator is negative, so $y > 0$.
- When $-2 < x < 1$, numerator is positive, denominator is negative, so $y < 0$.
- When $x > 1$, numerator is positive, denominator is positive, so $y > 0$.

So, the solution is $x \leq -2$ or $x > 1$, i.e., $x \in (-\infty, -2] \cup (1, \infty)$.

Solving an inequality that involves an absolute value

What about an inequality that involves an absolute value such as $|2x - 3| < 5$?

- $-5 < 2x - 3 < 5$
- $\frac{-5+3}{2} < x < \frac{5+3}{2}$
- $-1 < x < 4$, i.e., $x \in (-1, 4)$
- Check: $|2(1) - 3| < 5, |-1| < 5, 1 < 5$

Another example: $|4x - 6| > 2$?

- $4x - 6 < -2$ or $4x - 6 > 2$

- $x < \frac{-2+6}{4}$ or $x > \frac{2+6}{4}$
- $x < 1$ or $x > 2$, i.e., $x \in (-\infty, 1) \cup (2, \infty)$
- Check: $|4(-1) - 6| > 2, 10 > 2$ and $|4(3) - 6| > 2, 6 > 2$

Other examples

Find the values of x such that $5x^2 + 2x - 5 > x^2 + 2x + 4$.

- $5x^2 + 2x - 5 - x^2 - 2x - 4 > 0$
- $4x^2 - 9 > 0$
- $(2x + 3)(2x - 3) > 0$
- Solution: $x < -\frac{3}{2}$ or $x > \frac{3}{2}$, i.e., $x \in \left(-\infty, -\frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$.
- Check: $5(-2)^2 + 2(-2) - 5 > (-2)^2 + 2(-2) + 4, 11 > 4$.
- Check: $5(2)^2 + 2(2) - 5 > (2)^2 + 2(2) + 4, 19 > 12$.

Find the values of x such that $|-3x + 6| \leq 9$.

- $-9 \leq -3x + 6 \leq 9$
- $\frac{-9-6}{-3} \geq x \geq \frac{9-6}{-3}$
- $5 \geq x \geq -1$
- Solution: $-1 \leq x \leq 5$, i.e., $x \in [-1, 5]$.
- Check: $|-3(1) + 6| \leq 9, 3 \leq 9$.