8: Understanding Rational Exponents

Roots and radical expressions

If we raise a real number, a, to the power $1/n$, where n is a positive integer greater than 1, we obtain the *principal n-th root* of a: $a^{1/n} = \sqrt[n]{a}$. An expression containing such a root is called a *radical expression*.

- If $a > 0$, then $\sqrt[n]{a}$ is the positive real number b such that $b^n = a$. For example, $\sqrt[3]{8} = 2$ because $2^3 = 8$ and $\sqrt[2]{16} = \sqrt{16} = 4$ because $4^2 = 16$. Note that -4 is also a square root of 16 because $(-4)^2 = 16$, but it is not the principal square root.
- If $a < 0$ and n is odd, then $\sqrt[n]{a}$ is the negative real number b such that $b^n = a$. For example, $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$.
- If $a < 0$ and n is even, then $\sqrt[n]{a}$ is not a real number. For example, $\sqrt[2]{-16}$ is not a real number because there is no real number that we can square to equal –16.
- If $a = 0$, then $\sqrt[n]{a} = 0$.

More examples

- $\sqrt[3]{27} = 3$ because $3^3 = 27$.
- $\sqrt{36} = 6$ because $6^2 = 36$ (and -6 is also a square root of 36).
- $\sqrt[5]{-32} = -2$ because $(-2)^5 = -32$.
- $\sqrt[4]{\frac{1}{81}} = \frac{1}{3}$ because $\left(\frac{1}{3}\right)^4 = \frac{1}{3^4} = \frac{1}{81}$ (and $-\frac{1}{3}$ is also a fourth root of $\frac{1}{81}$).

Why it makes sense to define roots this way

It makes sense because $(a^{1/n})^n = \left(\sqrt[n]{a}\right)^n = a$. For example, $\left(\sqrt[3]{8}\right)^3 = 2^3 = 8$.

More general rational exponents

If we raise a real number, a, to the power m/n, where m is an integer and n is a positive integer greater than 1, we obtain the principal n-th root of a raised to the power m (in either order): $a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$. For example, $8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$ or $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$.

More examples

- $4^{3/2} = \left(\sqrt{4}\right)^3 = 2^3 = 8$ or $4^{3/2} = \sqrt{4^3} = \sqrt{64} = 8$.
- $(-8)^{2/3} = \left(\sqrt[3]{-8}\right)^2 = (-2)^2 = 4$ or $(-8)^{2/3} = \sqrt[3]{(-8)^2} = \sqrt[3]{64} = 4$.
- $81^{-3/4} = \frac{1}{81^{3/4}} = \frac{1}{\left(\sqrt[4]{81}\right)^3} = \frac{1}{3^3} = \frac{1}{27}$. (The other way is harder.)
- $\left(\frac{9}{16}\right)^{3/2} = \frac{9^{3/2}}{16^{3/2}} = \frac{\sqrt{9}^3}{\sqrt{16}^3} = \frac{3^3}{4^3} = \frac{27}{64}$. (The other way is harder.)

More complicated expressions with variables

Simplify the following expressions, ensuring that all exponents are positive:

- $(9x^2y^4)^{3/2} = \left(\sqrt{9x^2y^4}\right)^3 = (3xy^2)^3 = 27x^3y^6$.
- $\left(2x^{3/4}y^{1/2}\right)^8 = 2^8 \cdot \left(x^{3/4}\right)^8 \cdot \left(y^{1/2}\right)^8 = 256x^6y^4$.
- $\left(\frac{1}{64}x^3y^{-6}\right)^{1/3} = \left(\frac{1}{64}\right)^{1/3} \cdot (x^3)^{1/3} \cdot (y^{-6})^{1/3} = \frac{x}{4y^2}$.
- $\left(16x^{1/2}y^{-2}\right)^{3/4} = 16^{3/4} \cdot \left(x^{1/2}\right)^{3/4} \cdot (y^{-2})^{3/4} = \frac{8x^{3/8}}{y^{3/2}}$.
- $\left(\frac{x^{-4}y^2}{49}\right)^{-1/2} = \left(\frac{49}{x^{-4}y^2}\right)^{1/2} = 49^{1/2} \cdot \left(\frac{1}{x^{-4}}\right)^{1/2} \cdot \left(\frac{1}{y^2}\right)^{1/2} = \frac{7x^2}{y}$.

Rationalizing the denominator

Sometimes, we want any radical expressions to only be in the numerator. We can do this by *rationalizing the denominator* by multiplying by the appropriate fraction equivalent to 1. For example:

- $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.
- $\frac{1}{\sqrt[3]{2}} = \frac{1}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}} = \frac{\sqrt[3]{2^2}}{2} = \frac{\sqrt[3]{4}}{2}$.
- $\left(\frac{x}{2y}\right)^{1/3} = \frac{x^{1/3}}{(2y)^{1/3}} \cdot \frac{(2y)^{2/3}}{(2y)^{2/3}} = \frac{\left(4xy^2\right)^{1/3}}{2y} = \frac{\sqrt[3]{4xy^2}}{2y}$.